ON THE SAFE SET OF CARTESIAN PRODUCT OF TWO COMPLETE GRAPHS

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Abstract. For a connected graph G, a vertex subset S of $V(G)$ is a safe set if for every component C of the subgraph of G induced by S, $|C| \geq |D|$ holds for every component D of $G - S$ such that there exists an edge between C and D, and, in particular, if the subgraph induced by S is connected, then S is called a connected safe set. For a connected graph G, the safe number and the connected safe number of G are the minimum among sizes of the safe sets and the minimum among sizes of the connected safe sets, respectively, of G. Fujita *et al.* introduced these notions in connection with a variation of the facility location problem. In this paper, we study the safe number and the connected safe number of Cartesian product of two complete graphs. Figuring out a way to reduce the number of components to two without changing the size of safe set makes it sufficient to consider only partitions of an integer into two parts without which it would be much more complicated to take care of all the partitions. In this way, we could show that the safe number and the connected safe number of Cartesian product of two complete graphs are equal and present a polynomial-time algorithm to compute them. Especially, in the case where one of complete components has order at most four, we precisely formulate those numbers.

1. Introduction

Fujita *et al.* [2] introduced notions of safe set and connected safe set, motivated by the following problem. For a given topology of a building, it is required to place temporary accident refuges in addition to business spaces like discussion of conference rooms. Each temporary refuge should be available for the staff in every adjacent business space. (To mitigate the space cost, we assume that each temporary refuge will be used by the people in at most one of the adjacent business space.) Subject to the topology of the building being given, how can the temporary refuges be efficiently located so that the amount of business spaces is maximized? For more recent work on this subject, the reader may refer to Bapat *et al.* [1].

Given a graph G and a set X of vertices in G, we denote by $G[X]$ the subgraph of G induced by X. For a connected graph G, a set S of vertices in G is said to be a *safe set* if for every component C of $G[S]$, $|C| \geq |D|$ holds for every component D of $G - S$ such that there exists an edge between C and D, and, especially, if $G[S]$ is connected, then S is called a *connected*

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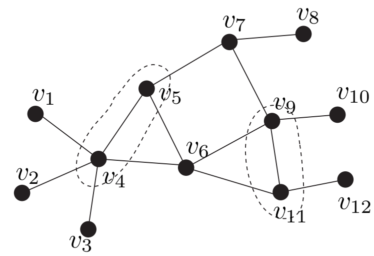

FIGURE 1. A set $\{v_4, v_5, v_9, v_{11}\}$ is a safe set. However, $S = \{v_3, v_4, v_9, v_{11}\}$ is not a safe set as $G - S$ has the component with 4 vertices v_5, v_6, v_7, v_8 even if each components of $G[S]$ has size two.

safe set. For a connected graph G, the *safe number* $s(G)$ of G is defined as $s(G) = \min\{|S| \mid S$ is a safe set of $G\}$, and the *connected safe number* $cs(G)$ of G is defined as $cs(G) = \min\{|S| \mid S$ is a connected safe set of $G\}$. See Figure 1 for an illustration. Fujita *et al.* [2] showed that for a graph G

$$s(G) \leq cs(G) \leq 2s(G) - 1$$

and any tree T with at most one vertex of degree at least three satisfies the equality $s(T) = cs(T)$. Other than this kind of trees, the complete graphs obviously satisfy the equality. In this regard, we thought that it would be interesting to study which graphs satisfy the equality and Cartesian products of complete graphs are good to start with.

The Cartesian product $G_1 \Box G_2$ of two simple graphs G_1 and G_2 is a graph with vertex set $V(G_1) \times V(G_2)$ and having two vertices (u_1, u_2) and (v_1, v_2) adjacent if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2, or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1.

By figuring out a way to reduce the number of components to two without changing the size of safe set, we shall show that for two integers $m, n \geq 1$, the safe number and the connected safe number of $K_m \Box K_n$ are the same, that is,

$$s(K_m \Box K_n) = cs(K_m \Box K_n),$$

and go further to compute the exact safe number. By symmetry, we assume $m \leq n$ without loss of generality. In addition, we mean by a component C of a graph G both the subgraph C and the vertex set C.

2. Main Results

We label the vertices of K_m as 1, 2, ..., m and K_n as 1, 2,..., n so that a vertex of G is denoted by (i, j) for some $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$.

If $m = 1$ or 2, then the safe number and the connected safe number can rather easily be computed:

Proposition 1. *For any positive integer n, the following are true:*

(i) $cs(K_1 \Box K_n) = s(K_1 \Box K_n) = \lceil \frac{n}{2} \rceil$;
(ii) $cs(K_2 \Box K_n) = s(K_2 \Box K_n) = n$.

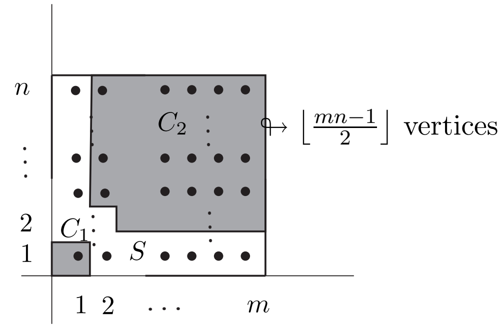

FIGURE 2. An illustration of the components of $K_m \square K_n - S$ mentioned in the proof of Proposition 2

Proof. We note that $K_1 \square K_n$ is the complete graph K_n. Thus $K_1 \square K_n - S$ is connected for a subset S of $V(K_1 \square K_n)$. Hence, by the definition of safe set, (i) is immediately true.

Now we show (ii). For simplicity, we let $G = K_2 \square K_n$. Suppose that $s(G) < n$ and let S be a minimum safe set of G. Then $|S| < n$, so $|V(G-S)| > n$. By the Pigeon Hole Principle, there exists j, $1 \le j \le n$, such that $(1,j) \in V(G-S)$ and $(2,j) \in V(G-S)$. Since $(1,j)$ and $(2,j)$ are adjacent and each vertex in G is adjacent to $(1,j)$ or $(2,j)$, $G-S$ is connected. However, $|V(G-S)| > |S|$, which contradicts the definition of safe set. Therefore $s(G) \ge n$. Since $\{(1,i) \mid 1 \le i \le n\}$ is a connected safe set of size n, we have

$$n \le s(G) \le cs(G) \le n$$

and so (ii) follows. □

From now on, we figure out the safe number and the connected safe number of $K_m \square K_n$ for $n \ge m \ge 3$. We denote by G the graph $K_m \square K_n$ for some integers $n \ge m \ge 3$ throughout this paper.

We first present the following useful proposition.

Proposition 2. *For $n \ge m \ge 3$, unless $m = n = 3$, the following are true:*

(i) *There exists a connected safe set of $K_m \square K_n$ of size $\lceil \frac{mn-1}{2} \rceil$ that is also a vertex cut;*
(ii) *There exists a minimum safe set of $K_m \square K_n$ that is a vertex cut.*

Proof. By the division algorithm, $\lfloor \frac{mn-1}{2} \rfloor = (n-1)q + r$ for some integers q, r with $0 \le r < n-1$. Obviously $q < m$. Since $n - 1 \le \lfloor \frac{mn-1}{2} \rfloor$ for $m \ge 3$, $q \ge 1$. Now we let C_2 be the subgraph of $K_m \square K_n$ induced by

$$\bigcup_{i=0}^{q-1} \{(m-i, 2), \ldots, (m-i, n)\} \cup \{(m-q, n-r+1), \ldots, (m-q, n)\}.$$

Then $|C_2| = (n-1)q + r = \lfloor \frac{mn-1}{2} \rfloor$. Now we take $(1,1)$ as a trivial subgraph C_1 as shown in Figure 2. Then obviously C_1 and C_2 are the components of $G - S$ where $S = V(G) - (V(C_1) \cup V(C_2))$. Moreover, since $n - r + 1 \ge 3$, $(m-q, 2) \in S$ and so $G[S]$ is connected. Thus S is a connected safe set. Since $|S| = \lceil \frac{mn-1}{2} \rceil$, $|C_1| \le |S|$, and $|C_2| \le |S|$. Thus S is a safe set of $K_m \square K_n$. Hence the safe number of $K_m \square K_n$ is less than or equal to $\lceil \frac{mn-1}{2} \rceil$ unless $m = n = 3$.

To show (ii), take a minimum safe set S of $K_m \Box K_n$ that is not a vertex cut. Then $G - S$ is connected. Let C be a component of $G[S]$ one of whose vertices is joined to a vertex in $G - S$. Then $mn - |S| \le |V(C)|$. Since $|V(C)| \le |S|$, $mn - |S| \le |S|$, or $|S| \ge \frac{mn}{2}$. Since $|S|$ is an integer, $|S| \ge \left\lceil \frac{mn}{2} \right\rceil$. By (i), there exists a safe set S^* with size $\left\lceil \frac{mn-1}{2} \right\rceil$ that is a vertex cut. If $\left\lceil \frac{mn-1}{2} \right\rceil < \left\lceil \frac{mn}{2} \right\rceil$, then S cannot be a minimum safe set since $|S^*| < |S|$. Therefore $\left\lceil \frac{mn-1}{2} \right\rceil = \left\lceil \frac{mn}{2} \right\rceil$. Since $|S^*| \le |S|$, S^* is a minimum safe set. $\square$

Let S be a vertex cut of G. By definition, any two vertices on the same row or any two vertices on the same column cannot be in distinct components in $G - S$. From this fact, we may make the following simple but very useful observation:

(§) If C is a component of $G - S$ and (i, j) is a vertex in C, then a vertex in the ith column or in the jth row belongs to either C or S.

Definition 3. Let $C_1, \dots, C_k$ be the components of $G - S$ for some vertex cut S of G. By the *component projection induced by S*, we mean the pair (Π_1, Π_2) of functions $\Pi_1 : [k] \to 2^{[m]}$ and $\Pi_2 : [k] \to 2^{[n]}$ defined as follows: for each $t \in [k]$,

$$\Pi_1(t) = \{i \mid (i, j) \in C_t\}, \qquad \Pi_2(t) = \{j \mid (i, j) \in C_t\}.$$

Since any two vertices on the same row or any two vertices on the same column cannot be in the same component as noted above,

$$\Pi_1(s) \cap \Pi_1(t) = \emptyset \text{ and } \Pi_2(s) \cap \Pi_2(t) = \emptyset$$

for distinct s, t in $[k]$. Thus we may assume that, for any vertex cut S of $K_m \Box K_n$, the components $C_1, \dots, C_k$ of $K_m \Box K_n - S$ satisfy the following properties throughout this paper:

$(*)$ (i) $(1, 1) \in \Pi_1(1) \times \Pi_2(1)$;
(ii) For $(i_1, j_1) \in C_t$ and $(i_2, j_2) \in C'_t$, $i_1 < i_2$ and $j_1 < j_2$ if and only if $t < t'$.

See Figure 3 for an illustration. Moreover, by definition,

$$\left| \bigcup_{t \in [k]} \Pi_1(t) \right| = \sum_{t=1}^{k} |\Pi_1(t)| \le m, \quad \left| \bigcup_{t \in [k]} \Pi_2(t) \right| = \sum_{t=1}^{k} |\Pi_2(t)| \le n. \tag{1}$$

In this paper, for a vertex cut S of $K_m \Box K_n$, we assume, unless otherwise mentioned, that the components of $K_m \Box K_n$ are arranged in this way.

By $(*)$, it can easily be checked that, for each $t \in [k]$,

$$\Pi_1(t) \times \Pi_2(t) = C_t \cup (S \cap (\Pi_1(t) \times \Pi_2(t)))$$

or

$$C_t = (\Pi_1(t) \times \Pi_2(t)) \setminus (S \cap (\Pi_1(t) \times \Pi_2(t))) . \tag{2}$$

For a graph G, we denote the number of components of G by $\omega(G)$.

Suppose that $\omega(G - S) \ge 3$. Then there exist two points (i, j) and (i', j) not in C_2 for i, i', j satisfying $\min \Pi_2(2) \le j \le \max \Pi_2(2)$, $i < \min \Pi_1(2)$, $\max \Pi_1(2) < i'$. Then, by the definition of $K_m \Box K_n$, (i, j) and (i', j) are joined to connect the two regions R^* and R_*. See Figure 4.

Therefore, in order for $G - S$ to be disconnected, $t = 2$. Now suppose that there exists a point (i, j) not in C_t for some $t \in \{1, 2\}$, $\min \Pi_1(1) \le i \le \max \Pi_1(t)$, $\min \Pi_2(t) \le j \le \max \Pi_2(t)$. We

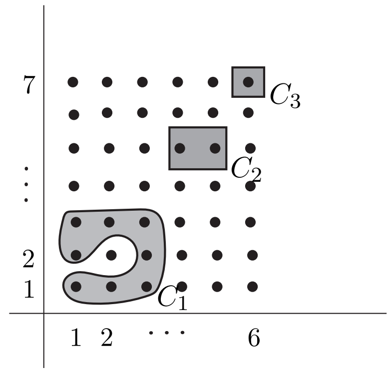

FIGURE 3. An illustration of components of $K_6 \square K_7 - S$ arranged so that $(i_1, j_1) \in C_t$ and $(i_2, j_2) \in C'_t$, $i_1 < i_2$ and $j_1 < j_2$ if and only if $t < t'$ where S is the set of vertices in the unshaded region. For these components, $\Pi_1(1) = \{1, 2, 3\}$, $\Pi_1(2) = \{4, 5\}$, $\Pi_1(3) = \{6\}$, $\Pi_2(1) = \{1, 2, 3\}$, $\Pi_2(2) = \{5\}$, $\Pi_2(3) = \{7\}$ where (Π_1, Π_2) is the component projection induced by S.

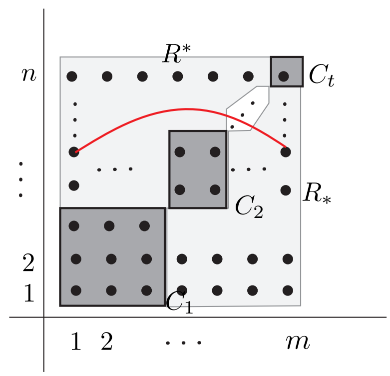

FIGURE 4. Two points connecting regions R^* and R_* when $t \geq 3$

assume $t = 1$. Then (i, j) is joined to (i, j') and (i', j) for $\max \Pi_1(1) < i'$ and $\max \Pi_2(1) < j'$ to join the two regions R^* and R_*. We obtain the same consequence even if $t = 2$. Thus we have shown that if $G - S$ is disconnected, then $t = 2$, $\Pi_1(1) \times \Pi_2(1) = C_1$, and $\Pi_1(2) \times \Pi_2(2) = C_2$. Therefore we obtain the following lemma:

Lemma 4. *Let $G = K_m \square K_n$ for some integers $m, n \geq 1$. Suppose that one of the following is true for a vertex cut S of G:*

(i) *either $\sum_{t=1}^{k} |\Pi_1(t)| < m$ or $\sum_{t=1}^{k} |\Pi_2(t)| < n$ where $k = \omega(G - S)$ and (Π_1, Π_2) is the component projection induced by S;*

(ii) *$\omega(G - S) \geq 3$.*

Then the subgraph $G[S]$ is connected.

We present a lemma which will play a key role throughout this paper.

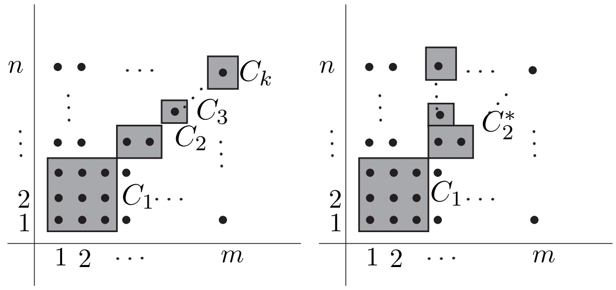

FIGURE 5. An illustration of forming C_2^* from C_2, …, C_k

Lemma 5. *Let $G = K_m \square K_n$ for some integers $n \geq m \geq 3$ with $n \geq 4$. Then there is a minimum safe set S^* of G satisfying $\omega(G - S^*) = 2$.*

Proof. Let S be a minimum safe set of G. By Proposition 2(ii), we may assume that S is a vertex cut. Therefore $\omega(G - S) \geq 2$. Suppose that $k := \omega(G - S) \geq 3$. Then $G[S]$ is connected by Lemma 4. Let C_1, …, C_k be the components of $G - S$ and (Π_1, Π_2) be the component projection induced by S. Suppose that $|\Pi_1(i)| \leq \lfloor \frac{m}{2} \rfloor$ for all $i \in [k]$. Since $|C_i| \leq \Pi_1(i)\Pi_2(i)$ for all i,

$$\sum_{i=1}^{k} |C_i| \leq \sum_{i=1}^{k} |\Pi_1(i)||\Pi_2(i)| \leq \left\lfloor \frac{m}{2} \right\rfloor \sum_{i=1}^{k} |\Pi_2(i)| = \left\lfloor \frac{m}{2} \right\rfloor n \leq \left\lfloor \frac{mn}{2} \right\rfloor.$$

Since $mn = |S| + \sum_{i=1}^{k} |C_i|$, $|S| = mn - \sum_{i=1}^{k} |C_i| \geq mn - \lfloor \frac{mn}{2} \rfloor = \lceil \frac{mn}{2} \rceil$. However, by Proposition 2(i), there is a safe set of size $\lfloor \frac{mn-1}{2} \rfloor$. Since $\lfloor \frac{mn-1}{2} \rfloor < \lceil \frac{mn}{2} \rceil$, S is not a minimum and we reach a contradiction. Thus there is $i \in [k]$ such that $\Pi_1(i) > \lfloor \frac{m}{2} \rfloor$, that is, $\Pi_1(i) \geq \lceil \frac{m}{2} \rceil$.

Without loss of generality, we may assume that $i = 1$. Let j_i^* denote the index of the leftmost column of vertices that belong to C_i for $i = 2$, …, k. Now we form the set C_2^* of vertices in the following way: Take the vertices of C_2. Then add a vertex in the ith row and the $(j - j_l^* + j_2^*)$th column whenever a vertex in the ith row and the jth column belongs to C_l for some $l \in \{3, \ldots, k\}$ as shown in Figure 5. In this way, we obtain a vertex cut S^* of G such that $G - S^*$ consists of two components C_1 and C_2^*. Then we can easily check that $|S| = |S^*|$ and $\omega(G - S^*) = 2$. Let (Π_1^*, Π_2^*) be the component projection induced by S^*. Then $|\Pi_1^*(1)| = |\Pi_1(1)|$, $|\Pi_2^*(1)| = |\Pi_2(1)|$, $|\Pi_1^*(2)| = |\Pi_1(2)|$, and $|\Pi_2^*(2)| = \max_{2 \leq j \leq k} |\Pi_2(j)|$. Furthermore, since $k \geq 3$, $|\Pi_2(1)| + \max_{2 \leq j \leq k} |\Pi_2(j)| < \sum_{i=1}^{k} |\Pi_2(i)| = n$. Thus $G[S^*]$ is connected by Lemma 4(i). Hence S^* is the only component the size of which is to be compared with $|C_i|$ for $i = 1$, …, k.

Now, since S is a connected safe set, $|C_1| \leq |S| = |S^*|$. To show that $|C_2^*| \leq |S^*|$, recall that $\Pi_1(1) \geq \lceil \frac{m}{2} \rceil$ and $\Pi_2(1) \geq \lceil \frac{n}{2} \rceil$ by our assumption. Thus $(m - |\Pi_1(1)|) \leq \lfloor \frac{m}{2} \rfloor \leq |\Pi_1(1)|$ and therefore

$$\begin{aligned} |C_2^*| &\leq (m - |\Pi_1(1)|) \left(\max_{2 \leq j \leq k} |\Pi_2(j)| \right) \leq (m - |\Pi_1(1)|) (n - \Pi_2(1)) \\ &\leq |\Pi_1(1)| (n - |\Pi_2(1)|) \leq |S^*|. \end{aligned}$$

Hence S^* is a connected safe set of size $|S|$. □

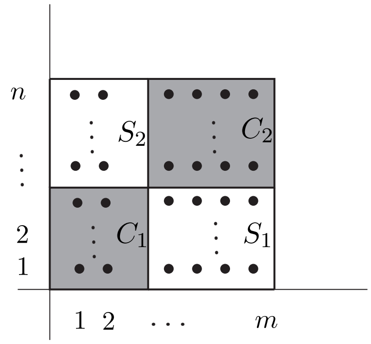

FIGURE 6. C_1 and C_2

Definition 6. Given integers $m \geq n > 1$, we let

$$P_2(m,n) = \{((m_1, m_2), (n_1, n_2)) \mid m_1 + m_2 = m, n_1 + n_2 = n, m_i, n_i \in \mathbb{N}\},$$

where $\mathbb{N}$ is the set of positive integers.

Lemma 7. *For any* $((m_1, m_2), (n_1, n_2)) \in P_2(m,n)$*, there is at most one* $j \in \{1,2\}$ *which satisfies*

$$mn - m_1n_1 - m_2n_2 < m_jn_j.$$

Proof. If $mn - m_1n_1 - m_2n_2 \geq m_tn_t$ for all $t \in \{1,2\}$, then we are done. Suppose that there is $j \in \{1,2\}$ such that $mn - m_1n_1 - m_2n_2 < m_jn_j$. Without loss of generality, we may assume $j = 1$, that is,

$$mn - m_1n_1 - m_2n_2 < m_1n_1$$

or

$$(m_1 + m_2)(n_1 + n_2) - m_1n_1 - m_2n_2 < m_1n_1.$$

We simplify the above inequality to obtain

$$m_1n_2 + (m_2 - m_1)n_1 < 0.$$

Since $m_1n_2 > 0$ and $n_1 > 0$, we have $m_1 > m_2$. Now

$$mn - m_1n_1 - m_2n_2 - m_2n_2 = (m_1 + m_2)(n_1 + n_2) - m_1n_1 - m_2n_2 - m_2n_2 = (m_1 - m_2)n_2 + m_2n_1.$$

Since $m_1 > m_2$, the right hand side of the second equality is positive. Therefore

$$mn - m_1n_1 - m_2n_2 > m_2n_2.$$

□

In Figure 6, suppose that $V(C_1) = [m_1] \times [n_1]$ and $V(C_2) = ([m] \setminus [m_1]) \times ([n] \setminus [n_1])$. Then the subgraph induced by $S := S_1 \cup S_2$ is not connected. By taking some vertices in C_1 or C_2 and adding them to S, we would like to obtain a connected safe set S^*. We denote the set of such vertices by Δ. Then $|S^*| = mn - m_1n_1 - m_2n_2 + |\Delta|$. If $mn - m_1n_1 - m_2n_2 \geq \max\{m_1n_1, m_2n_2\}$, then we add just one vertex as we wish to have S^* as small as possible. Otherwise, as long as S^* has at least $\max\{m_1n_1, m_2n_2\} - |\Delta|$, S^* is a safe set. That is, as long

as $|S^*| = mn - m_1n_1 - m_2n_2 + |\Delta| \geq \max\{m_1n_1, m_2n_2\} - |\Delta|$, S^* is a safe set. Solving this inequality for $|\Delta|$ gives

$$|\Delta| \geq \frac{\max\{m_1n_1, m_2n_2\} - mn + \sum_{i=1}^{2} m_in_i}{2}.$$

Motivated by this observation, we introduce the following notion.

Definition 8. Given integers $m, n \geq 3$, we define

$$\alpha(m,n) := \min\left\{mn - \sum_{i=1}^{2} m_in_i + \max\left\{\left\lceil\frac{\max\{m_1n_1, m_2n_2\} - mn + \sum_{i=1}^{2} m_in_i}{2}\right\rceil, 1\right\}\right\}$$

where the minimum is taken for each $((m_1, m_2), (n_1, n_2)) \in P_2(m, n)$.

Then the following is true.

Theorem 9. *Let $G = K_m \square K_n$ for some integers $m, n \geq 3$. Then*

$$cs(G) = \alpha(m,n).$$

Proof. Let S be a minimum connected safe set of G. By Lemma 5, we may assume that $\omega(G - S) = 2$. Let C_1 and C_2 be components of $G - S$. By (1),

$$|\Pi_1(1)| + |\Pi_1(2)| \leq m, \quad |\Pi_2(1)| + |\Pi_2(2)| \leq n$$

where (Π_1, Π_2) is the component projection induced by S. For notational convenience, let $|\Pi_1(1)| = m_1$, $m - |\Pi_1(1)| = m_2$, $|\Pi_2(1)| = n_1$, and $n - |\Pi_2(1)| = n_2$. Then

$$((m_1, m_2), (n_1, n_2)) \in P_2(m, n).$$

In addition, we define $r(t)$ in the following way:

$$r(1) = |S \cap (\Pi_1(1) \times \Pi_2(1))| \text{ and } r(2) = \left|S \cap \left[\,([m] \setminus \Pi_1(1)) \times ([n] \setminus \Pi_2(1))\,\right]\right|.$$

Then, by (2), $|C_1| = m_1n_1 - r(1)$. Furthermore,

$$\begin{aligned} C_2 &= \left[\,([m] \setminus \Pi_1(1)) \times ([n] \setminus \Pi_2(1))\,\right] \setminus S \\ &= \left[\,([m] \setminus \Pi_1(1)) \times ([n] \setminus \Pi_2(1))\,\right] \setminus \left[\,([m] \setminus \Pi_1(1)) \times ([n] \setminus \Pi_2(1))\,\right], \end{aligned}$$

so the equality $|C_2| = m_2n_2 - r(2)$ also holds. If $r(1) = r(2) = 0$, then S is disconnected by Lemma 4. Therefore one of $r(1)$ and $r(2)$ is at least 1. Note that if $\sum_{t=1}^{2} |\Pi_1(t)| = m$ and $\sum_{t=1}^{2} |\Pi_2(t)| = n$, then $[n] \setminus \Pi_2(1) = \Pi_1(2)$ and $[m] \setminus \Pi_1(1) = \Pi_2(2)$. Thus

$$\begin{aligned} |S| &= mn - \sum_{t=1}^{2} |C_t| = mn - (m_1n_1 - r(1)) - (m_2n_2 - r(2)) \\ &= mn - m_1n_1 - m_2n_2 + r(1) + r(2). \end{aligned} \tag{3}$$

By the definition of a connected safe set,

$$m_1n_1 - r(1) = |C_1| \leq |S| \quad \text{and} \quad m_2n_2 - r(2) = |C_2| \leq |S|.$$

Therefore

$$m_1n_1 - r(1) \leq mn - m_1n_1 - m_2n_2 + r(1) + r(2)$$

and

$$m_2n_2 - r(2) \leq mn - m_1n_1 - m_2n_2 + r(1) + r(2).$$

Then

$$m_1n_1 - (mn - m_1n_1 - m_2n_2) \le 2r(1) + r(2) \le 2(r(1) + r(2))$$

and

$$m_2n_2 - (mn - m_1n_1 - m_2n_2) \le r(1) + 2r(2) \le 2(r(1) + r(2)).$$

Therefore

$$\frac{1}{2}\left(\max\{m_1n_1, m_2n_2\} - (mn - m_1n_1 - m_2n_2)\right) \le r(1) + r(2).$$

Since $r(1)$ and $r(2)$ are integers,

$$\left\lceil \frac{\max\{m_1n_1, m_2n_2\} - (mn - m_1n_1 - m_2n_2)}{2} \right\rceil \le r(1) + r(2).$$

Furthermore, since one of $r(1)$ and $r(2)$ is at least 1,

$$\max\left\{ \left\lceil \frac{\max\{m_1n_1, m_2n_2\} - (mn - m_1n_1 - m_2n_2)}{2} \right\rceil, 1 \right\} \le r(1) + r(2).$$

Thus, by (3),

$$\begin{aligned} |S| &\ge mn - \sum_{i=1}^{2} m_in_i + \max\left\{ \left\lceil \frac{\max\{m_1n_1, m_2n_2\} - \left(mn - \sum_{i=1}^{2} m_in_i\right)}{2} \right\rceil, 1 \right\} \\ &\ge \alpha(m,n). \end{aligned}$$

Now we will show that there is a connected safe set with the size $\alpha(m,n)$.

Let $((m_1^*, m_2^*), (n_1^*, n_2^*))$ be an element of $P_2(m,n)$ that satisfies $\alpha(m,n)$, that is,

$$\alpha(m,n) = mn - \sum_{i=1}^{2} m_i^*n_i^* + \max\left\{ \left\lceil \frac{\max\{m_1^*n_1^*, m_2^*n_2^*\} - \left(mn - \sum_{i=1}^{2} m_i^*n_i^*\right)}{2} \right\rceil, 1 \right\}.$$

In addition, we let

$$D_1 = \{(i,j) \in V(G) \mid 1 \le i \le m_1^* \text{ and } 1 \le j \le n_1^*\}$$

and

$$D_2 = \{(i,j) \in V(G) \mid m_1^* + 1 \le i \le m \text{ and } n_1^* + 1 \le j \le n\}.$$

Then $|D_i| = m_i^*n_i^*$ for each $i = 1, 2$.

For simplicity, for each $t \in \{1,2\}$, let ν_t be a nonnegative integer such that

$$\nu_t = \left\lceil \frac{m_t^*n_t^* - \left(mn - \sum_{i=1}^{2} m_i^*n_i^*\right)}{2} \right\rceil.$$

Then, by Lemma 7, there is at most one ν_t such that $\nu_t \ge 1$. Without loss of generality, we may assume that $\nu_2 \le 0$.

Suppose that $\nu_1 \le 0$. Then we let

$$S = (V(G) \setminus (D_1 \cup D_2)) \cup \{(1,1)\}.$$

By definition, it is clear that S is connected and any component of $G-S$ is contained in D_1 or D_2. Furthermore,

$$\begin{aligned}|S| &= mn - \sum_{i=1}^{2} m_i^* n_i^* + 1 \\ &= mn - \sum_{i=1}^{2} m_i^* n_i^* + \max\left\{\left\lceil \frac{\max\{m_1^*n_1^*, m_2^*n_2^*\} - \left(mn - \sum_{i=1}^{2} m_i^* n_i^*\right)}{2}\right\rceil, 1\right\} \\ &= \alpha(m,n).\end{aligned}$$

Suppose that $G-S$ has two components C_1 and C_2 such that $C_t \subset D_t$ for each $t \in \{1,2\}$. Since $\nu_1 \le 0$, $m_1^*n_1^* \le mn - \sum_{i=1}^{2} m_i^* n_i^*$, it holds that

$$|C_1| \le |D_1| = m_1^* n_1^* \le mn - \sum_{i=1}^{2} m_i^* n_i^* \le \alpha(m,n) = |S|.$$

Similarly, since $\nu_2 \le 0$,

$$|C_2| \le |D_2| = m_2^* n_2^* \le mn - \sum_{i=1}^{2} m_i^* n_i^* \le \alpha(m,n) = |S|.$$

Therefore S is a safe set.

Now, suppose that $\nu_1 \ge 1$. By the division algorithm, there are integers q, r such that $\nu_1 = m_1 q + r$ with $0 \le r < m_1$. Let

$$D_1' = \bigcup_{i=1}^{q} \{(1,i), \dots, (m_1, i)\} \cup \{(1, q+1), \dots, (r, q+1)\}$$

where $\bigcup_{i=1}^{q}\{(1,i), \dots, (m_1,i)\} = \emptyset$ if $q = 0$. Note that $D_1' \subset D_1$ and $|D_1'| = \nu_1$. Let

$$S = (V(G) \setminus (D_1 \cup D_2)) \cup D_1'.$$

Then

$$|S| = mn - \sum_{i=1}^{2} m_i^* n_i^* + \nu_1.$$

Since $\nu_2 \le 0$ and $\nu_1 \ge 1$,

$$\nu_1 = \max\left\{\left\lceil \frac{\max\{m_1^*n_1^*, m_2^*n_2^*\} - \left(mn - \sum_{i=1}^{2} m_i^* n_i^*\right)}{2}\right\rceil, 1\right\}.$$

Thus

$$|S| = \alpha(m,n).$$

By its construction, it is clear that S is connected, one component of $G-S$ is contained in $D_1 \setminus D_1'$, and the other component is contained in D_2. Suppose that $G-S$ has two components C_1 and C_2 such that $C_1 \subset D_1 \setminus D_1'$ and $C_2 \subset D_2$. Since $\nu_2 \le 0$,

$$|C_2| \le |D_2| = m_2^* n_2^* \le mn - \sum_{i=1}^{2} m_i^* n_i^* \le |S|.$$

Now

$$\begin{aligned}|C_1| &\le |D_1 \setminus D_1'| = m_1^*n_1^* - \nu_1 \le (m_1^*n_1^* - 2\nu_1) + \nu_1 \\ &\le \left[m_1^*n_1^* - 2 \cdot \frac{m_1^*n_1^* - \left(mn - \sum_{i=1}^2 m_i^*n_i^*\right)}{2} \right] + \nu_1 = mn - \sum_{i=1}^2 m_i^*n_i^* + \nu_1 = |S|.\end{aligned}$$

Therefore S is a safe set. □

Theorem 10. *Let $G = K_m \square K_n$ for some $n \ge m \ge 1$. Then $s(G) = cs(G)$.*

Proof. By Proposition 1, it is sufficient to consider the cases $n \ge m \ge 3$. It is obvious that $s(G) \le cs(G)$. We show that $cs(G) \le s(G)$. By Proposition 2, there is a connected safe set of size $\lceil \frac{mn-1}{2} \rceil$ unless $m = n = 3$. It is easy to check that $\{(1,1),(1,2),(1,3),(2,1),(2,2)\}$ is a connected safe set of $K_3 \square K_3$. Therefore $cs(K_3 \square K_3) \le 5$. Thus $cs(G) \le \lceil \frac{mn}{2} \rceil$ for $n \ge m \ge 3$.

Let S be a minimum safe set of G. By Proposition 2, we may assume that S is a vertex cut. If $G[S]$ is connected, then we are done. Now suppose that S is not connected, that is, $\omega(G[S]) = t \ge 2$ for some nonnegative integer t. Then, by Lemma 4, $\omega(G - S) \le 2$. As we have shown that $cs(G) \le \lceil \frac{mn}{2} \rceil$, it is sufficient to show that $\lceil \frac{mn}{2} \rceil \le |S|$. Suppose that $\omega(G - S) = 1$. Since S is a safe set, $|G - S| \le |S|$. Then, since $|V(G)| = |G - S| + |S|$, $|V(G)| \le 2|S|$ or $\lceil \frac{mn}{2} \rceil \le |S|$.

Now suppose that $\omega(G - S) = 2$. Then, by Lemma 4, $|\Pi_1(1)| + |\Pi_1(2)| = m$, and $|\Pi_2(1)| + |\Pi_2(2)| = n$ where (Π_1, Π_2) is the component projection induced by S. Let C_1 and C_2 be the components of $G - S$. Then $G[S]$ has two components S_1 and S_2, and $|C_1| = |\Pi_1(1)||\Pi_2(1)|$, $|C_2| = |\Pi_1(2)||\Pi_2(2)|$, $|S_1| = |\Pi_1(1)||\Pi_2(2)|$, $|S_2| = |\Pi_1(2)||\Pi_2(1)|$. See Figure 6 for an illustration. Moreover, there are edges joining a vertex in C_1 and a vertex in S_1, a vertex in C_1 and a vertex in S_2, a vertex in C_2 and a vertex in S_1, a vertex in C_2 and a vertex in S_2, respectively.

Therefore, by the definition of a safe set,

$$|S_1| \ge \max\{|C_1|, |C_2|\} \quad \text{and} \quad |S_2| \ge \max\{|C_1|, |C_2|\}.$$

Then

$$\begin{aligned} mn &= (|\Pi_1(1)| + |\Pi_1(2)|)(|\Pi_2(1)| + |\Pi_2(2)|) \\ &= |\Pi_1(1)||\Pi_2(1)| + |\Pi_1(1)||\Pi_2(2)| + |\Pi_1(2)||\Pi_2(1)| + |\Pi_1(2)||\Pi_2(2)| \\ &= |C_1| + |S_1| + |S_2| + |C_2| \\ &\le 2(|S_1| + |S_2|) \le 2|S|, \end{aligned}$$

so $\frac{mn}{2} \le |S|$. Since $|S|$ is an integer, $\lceil \frac{mn}{2} \rceil \le |S|$ and we complete the proof. □

From Theorem 9 and Theorem 10, we immediately obtain our main result.

Theorem 11. *For two integers $m \ge n \ge 1$,*

$$s(K_m \square K_n) = cs(K_m \square K_n) = \alpha(m,n).$$

The following is an algorithm for MATLAB computing $\alpha(m,n)$ in a polynomial time.

safemin.m

```
function [y] = safemin(m,n)
if min(m,n) <3
    y=ceil(m*n/2)
else
    for i=1:(m-1)
            for j=1:(n-1)
        A(i,j) = m*n-i*j - (m-i)*(n-j) +  max(ceil((max(i*j,(m-i)*(n-j)) - (m*n-i*j - (m-i)*(n-j)))/2),1)
            end

    end
    y=min(min(A))
end
```

The following are the values of $\alpha(m,n)$ for $n,m \leq 10$ generated by the above algorithm.

	1	2	3	4	5	6	7	8	9	10
1	1	1	2	2	3	3	4	4	5	5
2	1	2	3	4	5	6	7	8	9	10
3	2	3	5	6	7	9	10	11	13	14
4	2	4	6	8	10	11	13	15	17	19
5	3	5	7	10	12	14	17	19	21	23
6	3	6	9	11	14	17	19	22	25	27
7	4	7	10	13	17	19	23	26	29	32
8	4	8	11	15	19	22	26	29	33	37
9	5	9	13	17	21	25	29	33	37	41
10	5	10	14	19	23	27	32	37	41	46

3. $\alpha(m,n)$ for $m \in \{3,4\}$, $n \geq m$

For $m \in \{3,4\}$, $n \geq m$, we precisely formulate $\alpha(m,n)$.

Theorem 12. *The safe number of $K_m \square K_n$ for $m \in \{3,4\}$, $n \geq m$ is as follows:*

(i) *If $m = 3$, then*

$$s(K_m \square K_n) = n + \left\lfloor \frac{n}{3} \right\rfloor + 1;$$

(ii) *If $m = 4$, then*

$$s(K_m \square K_n) = n + 4 \cdot \left\lfloor \frac{n}{5} \right\rfloor + \max\{i, 1\}$$

where $i \equiv n \pmod 5$ for some i, $0 \leq i \leq 4$.

Proof. We first show that upper bounds of $s(K_m \square K_n)$ are $n + \lfloor \frac{n}{3} \rfloor + 1$ and $n + 4 \cdot \lfloor \frac{n}{5} \rfloor + \max\{i, 1\}$, respectively, for $m = 3$ and $m = 4$ where $i \equiv n \pmod 5$ for some i, $0 \leq i \leq 4$.

Take $((m_1, m_2), (n_1, n_2)) = \left((1,2), \left(\lfloor \frac{n}{3} \rfloor, n - \lfloor \frac{n}{3} \rfloor\right)\right)$ from $P_2(3,n)$. Then

$$\sum_{i=1}^{2} m_i n_i = \left(\left\lfloor \frac{n}{3} \right\rfloor + 2\left(n - \left\lfloor \frac{n}{3} \right\rfloor\right)\right) = 2n - \left\lfloor \frac{n}{3} \right\rfloor,$$

$$\max\{m_1 n_1, m_2 n_2\} = 2\left(n - \left\lfloor \frac{n}{3} \right\rfloor\right),$$

and so

$$\Omega_3 := \max\left\{\left\lceil \frac{\max\{n_1, 2n_2\} - n - n_1}{2} \right\rceil, 1\right\} = \max\left\{\left\lceil \frac{2\left(n - \lfloor \frac{n}{3} \rfloor\right) - n - \lfloor \frac{n}{3} \rfloor}{2} \right\rceil, 1\right\} = 1.$$

Therefore

$$\alpha(3,n) = 3n - \sum_{i=1}^{2} m_i n_i + \Omega_3 = n + \left\lfloor \frac{n}{3} \right\rfloor + 1.$$

Hence

$$s(K_3 \square K_n) = \alpha(3,n) \leq n + \left\lfloor \frac{n}{3} \right\rfloor + 1$$

by Theorem 11.

Now we take $((m_1, m_2), (n_1, n_2)) = \left((1,3), \left(2 \cdot \left\lfloor \frac{n}{5} \right\rfloor, n - 2 \cdot \left\lfloor \frac{n}{5} \right\rfloor\right)\right)$ from $P_2(4,n)$. Suppose that $n = 5q + i$ for some i, $0 \leq i \leq 4$. Then

$$\sum_{i=1}^{2} m_i n_i = 2 \cdot \left\lfloor \frac{n}{5} \right\rfloor + 3\left(n - 2 \cdot \left\lfloor \frac{n}{5} \right\rfloor\right) = 3n - 4 \cdot \left\lfloor \frac{n}{5} \right\rfloor,$$

$$\max\{m_1 n_1, m_2 n_2\} = 3\left(n - 2 \cdot \left\lfloor \frac{n}{5} \right\rfloor\right),$$

and so

$$\begin{aligned} \Omega_4 &:= \max\left\{ \left\lceil \frac{\max\{m_1 n_1, m_2 n_2\} - 4n + \sum_{i=1}^{2} m_i n_i}{2} \right\rceil, 1 \right\} \\ &= \max\left\{ \left\lceil \frac{3\left(n - 2 \cdot \left\lfloor \frac{n}{5} \right\rfloor\right) - n - 4 \cdot \left\lfloor \frac{n}{5} \right\rfloor}{2} \right\rceil, 1 \right\} = \max\{i, 1\}. \end{aligned}$$

Therefore

$$\alpha(4,n) = 4n - \sum_{i=1}^{2} m_i n_i + \Omega_4 = n + 4 \cdot \left\lfloor \frac{n}{5} \right\rfloor + \max\{i, 1\}.$$

Hence

$$s(K_4 \square K_n) = \alpha(4,n) \leq n + 4 \cdot \left\lfloor \frac{n}{5} \right\rfloor + \max\{i, 1\}$$

by Theorem 11.

Now we show that $\alpha(3,n) \geq n + \left\lfloor \frac{n}{3} \right\rfloor + 1$ for $n \geq 4$. Let $((m_1, m_2), (n_1, n_2)) \in P_2(3,n)$ be a partition by which $\alpha(3,n)$ is achieved. That is,

$$\alpha(3,n) = 3n - \sum_{i=1}^{2} m_i n_i + \max\left\{ \left\lceil \frac{\max\{n_1, 2n_2\} - n - n_1}{2} \right\rceil, 1 \right\}.$$

Without loss of generality, we may assume that $m_1 = 1$ and $m_2 = 2$. Then

$$3n - \sum_{i=1}^{2} m_i n_i = 3n - n_1 - 2n_2 = n + n_1,$$

so

$$\Omega_3 := \max\left\{ \left\lceil \frac{\max\{n_1, 2n_2\} - n - n_1}{2} \right\rceil, 1 \right\} \quad \text{and} \quad \alpha(3,n) = n + n_1 + \Omega_3.$$

Suppose $n_1 \geq 2n_2$. Then $\max\{n_1, 2n_2\} - n - n_1 = -n < 0$, so $\Omega_3 = 1$. Thus $\alpha(3,n) = n + n_1 + 1$. On the other hand, since $n_1 + n_2 = n$, $n_1 \geq 2n_2$ implies $2n_2 \geq n$. Then

$$\alpha(3,n) = n + n_1 + 1 \geq n + \frac{n}{2} + 1 > n + \left\lfloor \frac{n}{3} \right\rfloor + 1,$$

and we reach a contradiction as we have shown that $\alpha(3,n) \leq n + \lfloor \frac{n}{3} \rfloor + 1$. Thus $n_1 < 2n_2$ and so $\max\{n_1, 2n_2\} = 2n_2$. Then

$$2n_2 - n - n_1 = 2(n - n_1) - n - n_1 = n - 3n_1,$$

so $\Omega_3 = \max\left\{\lceil \frac{n-3n_1}{2} \rceil, 1\right\}$.

Suppose that $n - 3n_1 < 2$. Then $n_1 > \frac{n-2}{3}$ and $\Omega_3 = 1$. Since n_1 is an integer, $n_1 > \lfloor \frac{n}{3} \rfloor$. Therefore

$$\alpha(3,n) = n + n_1 + 1 > n + \left\lfloor \frac{n}{3} \right\rfloor + 1$$

and we reach a contradiction. Therefore $n - 3n_1 \geq 2$ and so $\Omega_3 = \lceil \frac{n-3n_1}{2} \rceil$. Thus

$$\alpha(3,n) = n + n_1 + \left\lceil \frac{n - 3n_1}{2} \right\rceil = n + \left\lceil \frac{n - n_1}{2} \right\rceil \geq n + \left\lceil \frac{n+1}{3} \right\rceil.$$

Since $\lceil \frac{n+1}{3} \rceil \geq \lfloor \frac{n}{3} \rfloor + 1$ for $n \geq 4$, we obtain $\alpha(3,n) \geq n + \lfloor \frac{n}{3} \rfloor + 1$ from the above inequality. Since $\alpha(3,n) \leq n + \lfloor \frac{n}{3} \rfloor + 1$, we conclude that $\alpha(3,n) = n + \lfloor \frac{n}{3} \rfloor + 1$.

In the following, we will show that $n + 4 \cdot \lfloor \frac{n}{5} \rfloor + \max\{i, 1\} \leq \alpha(4,n)$ where $i \equiv n \pmod 5$ for some i, $0 \leq i \leq 4$.

Let $((m_1, m_2), (n_1, n_2)) \in P_2(4,n)$ be a partition by which $\alpha(4,n)$ is achieved, that is,

$$\alpha(4,n) = 4n - \sum_{i=1}^{2} m_i n_i + \max\left\{\left\lceil \frac{\max\{m_1 n_1, m_2 n_2\} - n - 2n_1}{2} \right\rceil, 1\right\}.$$

Then either $\{m_1, m_2\} = \{2\}$ or $\{m_1, m_2\} = \{1, 3\}$. Assume $\{m_1, m_2\} = \{2\}$. Then

$$4n - \sum_{i=1}^{2} m_i n_i = 4n - 2n_1 - 2n_2 = 2n.$$

Since $\max\left\{\left\lceil \frac{\max\{2n_1, 2n_2\} - n - 2n_1}{2} \right\rceil, 1\right\} \geq 1$, $\alpha(4,n) \geq 2n + 1$. However, we have already shown that $\alpha(4,n) \leq n + 4 \cdot \lfloor \frac{n}{5} \rfloor + \max\{i, 1\}$, which is less than $2n + 1$, and we reach a contradiction. Thus $\{m_1, m_2\} = \{1, 3\}$. Without loss of generality, we may assume that $m_1 = 1$ and $m_2 = 3$. Then

$$4n - \sum_{i=1}^{2} m_i n_i = 4n - n_1 - 3n_2 = n + 2n_1$$

and

$$\Omega_4 := \max\left\{\left\lceil \frac{\max\{n_1, 3n_2\} - n - 2n_1}{2} \right\rceil, 1\right\}.$$

If $n_1 \geq 3n_2$, then $\max\{n_1, 3n_2\} - n - 2n_1 = -n - n_1 < 0$, a contradiction. Therefore $3n_2 \geq n_1$ and so $\Omega_4 = \max\left\{\lceil \frac{2n-5n_1}{2} \rceil, 1\right\}$ since $n_1 + n_2 = n$. If $2n - 5n_1 < 2$, then $\Omega_4 = 1$ and so

$$\alpha(4,n) = n + 2n_1 + 1 > n + 4 \cdot \left\lfloor \frac{n}{5} \right\rfloor + \max\{i, 1\},$$

which is a contradiction. Thus $2n - 5n_1 \geq 2$. Then $n_1 \leq \frac{2n-2}{5}$ and $\Omega_4 = \lceil \frac{2n-5n_1}{2} \rceil$, so

$$\alpha(4,n) = n + 2n_1 + \left\lceil \frac{2n - 5n_1}{2} \right\rceil = n + \left\lceil \frac{2n - n_1}{2} \right\rceil \geq n + \left\lceil \frac{1 + 4n}{5} \right\rceil.$$

Since $i \equiv n \pmod 5$, $n = 5k + i$ for some integer k. Thus

$$\left\lceil \frac{1+4n}{5} \right\rceil = \left\lceil \frac{1+20k+4i}{5} \right\rceil = 4k + \left\lceil \frac{1+4i}{5} \right\rceil \geq 4 \cdot \left\lfloor \frac{n}{5} \right\rfloor + \max\{i, 1\}$$

and so $\alpha(4, n) \geq n + 4 \cdot \left\lfloor \frac{n}{5} \right\rfloor + \max\{i, 1\}$. Thus $\alpha(4, n) = n + 4 \cdot \left\lfloor \frac{n}{5} \right\rfloor + \max\{i, 1\}$ and we complete the proof. $\square$

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